Chapter 4
Representations of the Electromagnetic Field

Abstract A full description of the electromagnetic field requires a quantum statistical treatment. The electromagnetic field has an infinite number of modes and each mode requires a statistical description in terms of its allowed quantum states. However, as the modes are described by independent Hilbert spaces, we may form the statistical description of the entire field as the product of the distribution function for each mode. This enables us to confine our description to a single mode without loss of generality.

In this chapter we introduce a number of possible representations for the density operator of the electromagnetic field. One representation is to expand the density operator in terms of the number states. Alternatively the coherent states allow a number of possible representations via the P function, the Wigner function and the Q function.

4.1 Expansion in Number States

The number or Fock states form a complete set, hence a general expansion of ρ is

$$\rho = \sum C_{nm}|n\rangle\langle m| \ . \tag{4.1}$$

The expansion coefficients C_{nm} are complex and there is an infinite number of them. This makes the general expansion rather less useful, particularly for problems where the phase-dependent properties of the electromagnetic field are important and hence the full expansion is necessary. However, in certain cases where only the photon number distribution is of interest the reduced expansion

$$\rho = \sum P_n|n\rangle\langle n| \ , \tag{4.2}$$

may be used. Here P_n is a probability distribution giving the probability of having n photons in the mode. This is not a general representation for all fields but may prove useful for certain fields. For example, a chaotic field, which has no phase information, has the distribution

$$P_n = \frac{1}{(1+\bar{n})}\left(\frac{\bar{n}}{1+\bar{n}}\right)^n , \tag{4.3}$$

where $\bar{n}$ is the mean number of photons. This is derived by maximising the entropy

$$S = -\mathrm{TR}\{\rho \ln \rho\} , \tag{4.4}$$

subject to the constraint $\mathrm{Tr}\{\rho a^\dagger a\} = \bar{n}$, and is just the usual Planck distribution for black-body radiation with

$$\bar{n} = \frac{1}{\mathrm{e}^{\hbar\omega/kT} - 1} . \tag{4.5}$$

The second-order correlation function $g^2(0)$ may be written according to (3.66)

$$g^2(0) = 1 + \frac{V(n) - \bar{n}}{\bar{n}^2} , \tag{4.6}$$

where $V(n)$ is the variance of the distribution function P_n. Hence, for the power-law distribution $V(n) = \bar{n}^2 + \bar{n}$ we find $g^{(2)}(0) = 2$. For a field with a Poisson distribution of photons

$$P_n = \frac{e^{-\bar{n}}}{n!}\bar{n}^n \tag{4.7}$$

the variance $V(n) = \bar{n}$, hence $g^{(2)}(0) = 1$.

A coherent state has a Poisson distribution of photons. However, a measurement of $g^{(2)}(0)$ would not distinguish between a coherent state and a field prepared from an incoherent mixture with a Poisson distribution. In order to distinguish between these two fields a phase-dependent measurement such as a measurement of ΔX_1, ΔX_2 would need to be made.

4.2 Expansion in Coherent States

4.2.1 P Representation

The coherent states $|\alpha\rangle$ form a complete set of states, in fact, an overcomplete set of states. They may therefore be used as a basis set despite the fact that they are non-orthogonal. The following diagonal representation in terms of coherent states was introduced independently by *Glauber* [1] and *Sudarshan* [2]

$$\rho = \int P(\alpha)|\alpha\rangle\langle\alpha| \mathrm{d}^2\alpha , \tag{4.8}$$

where $\mathrm{d}^2\alpha = \mathrm{d}(\mathrm{Re}\{\alpha\})\mathrm{d}(\mathrm{Im}\{\alpha\})$. It has found wide-spread application in quantum optics.

Now it might be imagined that the function $P(\alpha)$ is analogous to a probability distribution for the values of α. However, in general this is not the case since the projection operator $|\alpha\rangle\langle\alpha|$ is onto non-orthogonal states, and hence $P(\alpha)$ cannot be interpreted as a genuine probability distribution. We may note that the coherent states $|\alpha\rangle$ and $|\alpha'\rangle$ are approximately orthogonal for $|\alpha-\alpha'|\gg 1$, see (2.41). Hence, if $P(\alpha)$ is slowly varying over such large ranges of the parameter there is an approximate sense in which $P(\alpha)$ may be interpreted with a classical description. There are, however, certain quantum states of the radiation field where $P(\alpha)$ may take on negative values or become highly singular. For these fields there is no classical description and $P(\alpha)$ clearly cannot be interpreted as a classical probability distribution. Let us now consider examples of fields which may be described by the P representation:

(i) Coherent state

If

$$\rho = |\alpha_0\rangle\langle\alpha_0| \,, \tag{4.9}$$

then

$$P(\alpha) = \delta^{(2)}(\alpha-\alpha_0) \,. \tag{4.10}$$

(ii) Chaotic state

For a chaotic state it follows from the central limit theorem that $P(\alpha)$ is a Gaussian

$$P(\alpha) = \frac{1}{\pi\bar{n}}\mathrm{e}^{-|\alpha|^2/\bar{n}} \,. \tag{4.11}$$

That this is equivalent to the result for P_n is clear if we take matrix elements

$$P_n = \langle n|\rho|n\rangle = \int P(\alpha)|\langle n|\alpha\rangle|^2 \mathrm{d}^2\alpha = \frac{1}{\pi\bar{n}}\int \mathrm{e}^{-|\alpha|^2/\bar{n}}\frac{|\alpha|^{2n}}{n!}\mathrm{e}^{-|\alpha|^2}\mathrm{d}^2\alpha \,. \tag{4.12}$$

Using the identity

$$\pi^{-1}(l!m!)^{-1/2}\int \exp(-C|\alpha|^2)\alpha^l(\alpha^*)^m \mathrm{d}^2\alpha = \delta_{lm}C^{-(m+1)} \,, \tag{4.13}$$

which holds for $C>0$ and choosing

$$C = \frac{1+\bar{n}}{\bar{n}}$$

we find

$$P_n = \frac{1}{1+\bar{n}}\left(\frac{\bar{n}}{1+\bar{n}}\right)^n \,. \tag{4.14}$$

For a mixture of a coherent and a chaotic state the P function is

$$P(\alpha) = \frac{1}{\pi\bar{n}}\mathrm{e}^{-|\alpha-\alpha_0|^2/\bar{n}} \,, \tag{4.15}$$

which may be derived using the following convolution property of $P(\alpha)$. Consider a field produced by two independent sources. The first source acting constructs the field

$$\rho_1 = \int P_1(\alpha_1)|\alpha_1\rangle\langle\alpha_1|\mathrm{d}^2\alpha_1 \,. \tag{4.16}$$

Acting alone the second source would produce the field

$$\rho_2 = \int P_2(\alpha_2)|\alpha_2\rangle\langle\alpha_2|\mathrm{d}^2\alpha_2 = \int P_2(\alpha_2)D(\alpha_2)|0\rangle\langle 0|D^{-1}(\alpha_2)\mathrm{d}^2\alpha_2 \,. \tag{4.17}$$

The second source acting after the first field generates the field

$$\begin{aligned}\rho &= \int P_2(\alpha_2)D(\alpha_2)\rho_1 D^{-1}(\alpha_2)\mathrm{d}^2\alpha_2 \\ &= \int P_2(\alpha_2)P_1(\alpha_1)|\alpha_1+\alpha_2\rangle\langle\alpha_1+\alpha_2|\mathrm{d}^2\alpha_1\,\mathrm{d}^2\alpha_2 \,. \end{aligned} \tag{4.18}$$

The weight function $P(\alpha)$ for the superposed excitations is therefore

$$\begin{aligned}P(\alpha) &= \int \delta^2(\alpha-\alpha_1-\alpha_2)P_1(\alpha_1)P_2(\alpha_2)\mathrm{d}^2\alpha_1\,\mathrm{d}^2\alpha_2 \\ &= \int P_1(\alpha-\alpha')P_2(\alpha')\mathrm{d}^2\alpha' \,. \end{aligned} \tag{4.19}$$

We see that the distribution function for the superposition of two fields is the convolution of the distribution functions for each field.

a) Correlation Functions

The $P(\alpha)$ representation is convenient for evaluating normally-ordered products of operators, for example

$$\langle a^{\dagger n}a^m\rangle = \int P(\alpha)\alpha^{*n}\alpha^m\mathrm{d}^2\alpha \,. \tag{4.20}$$

This reduces the taking of quantum mechanical expectation values to a form similar to classical averaging.

Let us express the second-order correlation function in terms of $P(\alpha)$

$$g^{(2)}(0) = 1 + \frac{\int P(\alpha)[(|\alpha|^2) - \langle|\alpha|^2\rangle]^2\,\mathrm{d}^2\alpha}{[\int P(\alpha)|\alpha|^2\,\mathrm{d}^2\alpha]^2} \,. \tag{4.21}$$

This looks functionally identical to the expression for classical fields. However, the argument that $g^{(2)}(0)$ must be greater than or equal to unity no longer holds since for certain fields as we have mentioned $P(\alpha)$ may take on negative values and allow for a $g^{(2)}(0) < 1$, that is, photon antibunching.

A similar result may be derived for the squeezing. We may write the variances in X_1 and X_2 as

$$\begin{aligned}\Delta X_1^2 &= \{1 + \int P(\alpha)[(\alpha+\alpha^*) - (\langle\alpha\rangle + \langle\alpha^*\rangle)]^2\,\mathrm{d}^2\alpha\}, \\ \Delta X_2^2 &= \left\{1 + \int P(\alpha)\left[\left(\frac{\alpha-\alpha^*}{\mathrm{i}}\right) - \left(\frac{\langle\alpha\rangle - \langle\alpha^*\rangle}{\mathrm{i}}\right)\right]^2\,\mathrm{d}^2\alpha\right\} \,. \end{aligned} \tag{4.22}$$

The condition for squeezing $\Delta X_1^2 < 1$, requires that $P(\alpha)$ takes on a negative value along either the real or imaginary axis in the complex plane, but not both simultaneously. Thus squeezing and antibunching are phenomena which are the exclusive property of quantum fields and may not be generated by classical fields. Some ambiguity may arise in the case of squeezing which only has significance for quantum fields. If a classical field is assumed from the outset arbitrary squeezing may occur in either quadrature or both simultaneously.

Quantised fields for which $P(\alpha)$ is a positive function do not exhibit quantum properties such as photon antibunching and squeezing. Such fields may be simulated by a classical description which treats the complex field amplitude ε as a stochastic random variable with the probability distribution $P(\varepsilon)$ and hence may be considered as quasiclassical. Coherent and chaotic fields are familiar examples of fields with a positive P representation. Quantum fields exhibiting antibunching and/or squeezing cannot be described in classical terms. For such fields the P representation may be negative and highly singular. The coherent state has a P representation which is a delta function, defining the boundary between quantum and classical behaviour. For fields exhibiting quantum behaviour such as a number state $|n\rangle$ or squeezed state $|\alpha,\varepsilon\rangle$ no representation for $P(\alpha)$ in terms of tempered distributions exists. Though representations in terms of generalised functions do exist [3], such representations are highly singular, for example, derivatives of delta functions. We shall therefore look for alternative representations to describe such quantum fields.

b) *Covariance Matrix*

Gaussian processes which arise, for example, in linearized fluctuation theory may be characterized by a covariance matrix. A covariance matrix may be defined by

$$C(a,a^\dagger) = \begin{pmatrix} \langle a^2\rangle - \langle a\rangle^2 & \frac{1}{2}\langle aa^\dagger + a^\dagger a\rangle - \langle a^\dagger\rangle\langle a\rangle \\ \frac{1}{2}\langle aa^\dagger + a^\dagger a\rangle - \langle a^\dagger\rangle\langle a\rangle & \langle a^{\dagger 2}\rangle - \langle a^\dagger\rangle^2 \end{pmatrix} . \tag{4.23}$$

One may also introduce a correlation matrix $C(X_1,\ X_2)$ for the quadrature phase operators X_1 and X_2:

$$C(X_1,X_2)_{p,q} = \frac{1}{2}\langle X_p X_q + X_q X_p\rangle - \langle X_p\rangle\langle X_q\rangle \ (p,q = 1,2) . \tag{4.24}$$

These two correlation matrices are related by

$$C(X_1,X_2) = \Omega C(a,a^\dagger)\Omega^T \tag{4.25}$$

where

$$\Omega = \begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix} .$$

The covariance matrix $C_p(\alpha,\ \alpha^*)$ defined by the moments of α and α^* over $P(\alpha,\ \alpha^*)$ is related to the covariance matrix $C(a,\ a^\dagger)$ by

$$C(a,a^\dagger) = C_p(\alpha,\alpha^*) + \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{4.26}$$

The distribution can be written in terms of the real variables

$$x_1 = \alpha + \alpha^*, \quad x_2 = \frac{1}{\mathrm{i}}(\alpha - \alpha^*)$$

for which the covariance matrix relation is

$$C(X_1, X_2) = C_p(x_1, x_2) + I \,. \tag{4.27}$$

c) Characteristic Function

In practice, it proves useful to evaluate the P function through a characteristic function.

The density operator ρ is uniquely determined by its characteristic function

$$\chi(\eta) = \mathrm{Tr}\{\rho \mathrm{e}^{\eta a^\dagger - \eta^* a}\} \,.$$

We may also define normally and antinormally ordered characteristic functions

$$\chi_{\mathrm{N}}(\eta) = \mathrm{Tr}\{\rho \mathrm{e}^{\eta a^\dagger} \mathrm{e}^{-\eta^* a}\} \,, \tag{4.28}$$

$$\chi_{\mathrm{A}}(\eta) = \mathrm{Tr}\{\rho \mathrm{e}^{-\eta^* a} \mathrm{e}^{\eta a^\dagger}\} \,. \tag{4.29}$$

Using the relation (2.25) the characteristic functions are related by

$$\chi(\eta) = \chi_{\mathrm{N}}(\eta) \exp\left(-\frac{1}{2}|\eta|^2\right) \,. \tag{4.30}$$

If the density operator ρ has a P representation, then $\chi_{\mathrm{N}}(\eta)$ is given by

$$\chi_{\mathrm{N}}(\eta) = \int \langle\alpha|\mathrm{e}^{\eta a^\dagger} \mathrm{e}^{-\eta^* a}|\alpha\rangle P(\alpha) \mathrm{d}^2\alpha = \int \mathrm{e}^{\eta\alpha^* - \eta^*\alpha} P(\alpha) \mathrm{d}^2\alpha \,. \tag{4.31}$$

Writing η and α in terms of their real and imaginary parts we find that (4.31) expresses $\chi_{\mathrm{N}}(\eta)$ as a two-dimensional Fourier transform of $P(\alpha)$. The solution for $P(\alpha)$ is the inverse Fourier transform

$$P(\alpha) = \frac{1}{\pi^2} \int \mathrm{e}^{\alpha\eta^* - \alpha^*\eta} \chi_{\mathrm{N}}(\eta) \mathrm{d}^2\eta \,. \tag{4.32}$$

Thus the criterion for the existence of a P representation is the existence of a Fourier transform for the normally-ordered characteristic function $\chi_{\mathrm{N}}(\eta)$.

4.2.2 Wigner's Phase-Space Density

The first quasi-probability distribution was introduced into quantum mechanics by *Wigner* [4]. The Wigner function may be defined as the Fourier transform of the symmetrically ordered characteristic function $\chi(\eta)$

$$W(\alpha) = \frac{1}{\pi^2} \int \exp(\eta^* \alpha - \eta \alpha^*) \chi(\eta) \mathrm{d}^2\eta \ . \tag{4.33}$$

The Wigner distribution always exists but is not necessarily positive.

The relationship between the Wigner distribution and the $P(\alpha)$ distribution may be obtained via the characteristic functions. Using (4.30) we may express the Wigner function as

$$\begin{aligned} W(\alpha) &= \frac{1}{\pi^2} \int \exp(\eta^* \alpha - \eta \alpha^*) \chi_{\mathrm{N}}(\eta) \mathrm{e}^{-1/2|\eta|^2} \mathrm{d}^2\eta \\ &= \frac{1}{\pi^2} \int \mathrm{Tr}\{\rho \mathrm{e}^{\eta(a^\dagger - \alpha^*)} \mathrm{e}^{-\eta^*(a-\alpha)}\} \mathrm{e}^{-1/2|\eta|^2} \mathrm{d}^2\eta \\ &= \frac{1}{\pi^2} \int P(\beta) \exp[\eta(\beta^* - \alpha^*) - \eta^*(\beta - \alpha) - \frac{1}{2}|\eta|^2] \mathrm{d}^2\eta \ \mathrm{d}^2\beta \ . \end{aligned} \tag{4.34}$$

Substituting $\varepsilon = \eta/\sqrt{2}$ leads to

$$W(\alpha) = \frac{2}{\pi^2} \int P(\beta) \exp[\sqrt{2}\varepsilon(\beta^* - \alpha^*) - \sqrt{2}\varepsilon^*(\beta - \alpha) - |\varepsilon|^2] \mathrm{d}^2\varepsilon \ \mathrm{d}^2\beta \ . \tag{4.35}$$

The integral may be evaluated using the identity

$$\frac{1}{\pi} \int \mathrm{d}^2\eta \exp(-\lambda|\eta|^2 + \mu\eta + \nu\eta^*) = \frac{1}{\lambda} \exp\left(\frac{\mu\nu}{\lambda}\right) \tag{4.36}$$

which holds for $\mathrm{Re}\{\lambda\} > 0$ and arbitrary μ, ν. This gives

$$W(\alpha) = \frac{2}{\pi} \int P(\beta) \exp(-2|\beta - \alpha|^2) \mathrm{d}^2\beta \ . \tag{4.37}$$

That is, the Wigner function is a Gaussian convolution of the P function.

The covariance matrix $C_w(\alpha,\ \alpha^*)$ defined by the moments of α and α^* over $W(\alpha,\ \alpha^*)$ is related to the covariance matrix $C(a,\ a^\dagger)$ defined by (4.23) by

$$C(a, a^\dagger) = C_w(\alpha, \alpha^*) \ . \tag{4.38}$$

The error areas discussed in Chap. 2 may rigorously be derived as contours of the Wigner function. We shall now study the Wigner functions for several states of this radiation field and their corresponding contours.

As it is defined in (4.33) the Wigner function is normalised as

$$\int \mathrm{d}^2\alpha W(\alpha) = 1 \tag{4.39}$$

We have defined the quadrature phase operators X_i as (2.56)

$$a = \frac{1}{2}(X_1 + \mathrm{i}X_2) \tag{4.40}$$

If we denote the eigenstates of X_i as $|x_i\rangle$, we can write the Wigner function as a function of x_i using

$$W(x_1,x_2) = \frac{1}{4} W(\alpha)\,|_{\alpha=(x_1+\mathrm{i}x_2)/2} \tag{4.41}$$

so that the marginal distributions of W,

$$P(x_k) = \int_{-\infty}^{\infty} \mathrm{d}x_{\bar{k}} W(x_1,x_2) \tag{4.42}$$

are given by $P(x_i) = \langle x_i|\rho|x_i\rangle$, where $\bar{k} = k - (-1)^k$.

For certain states of the radiation field the Wigner function may be written in the Gaussian form

$$W(x_1,x_2) = N \exp\left(-\frac{1}{2}Q\right) \tag{4.43}$$

where Q is the quadratic form

$$Q = (\boldsymbol{x}-\boldsymbol{a})^{\mathrm{T}} A^{-1} (\boldsymbol{x}-\boldsymbol{a}) \tag{4.44}$$

and N is the normalization. A contour of the Wigner function is the curve $Q = 1$. We choose to work in the phase space where x_1 and x_2 are the c-number variables corresponding to the quadrature phase amplitudes X_1 and X_2.

a) *Coherent State*

For a coherent state $|\alpha\rangle = |\frac{1}{2}(X_1 + \mathrm{i}X_2)\rangle$ the Wigner function is

$$W(x_1,x_2) = \frac{1}{2\pi} \exp\left[-\frac{1}{2}(x_1'^2 + x_2'^2)\right] \tag{4.45}$$

where $x_i' = x_i - X_i$. The contour of the Wigner function given by $Q = 1$ is

$$x_1'^2 + x_2'^2 = 1\ . \tag{4.46}$$

Thus the error area is a circle with radius 1 centred on the point $(X_1,\ X_2)$ (Fig. 2.1a).

b) *Squeezed State*

The Wigner function for a squeezed state is

$$W(x_1,x_2) = \frac{1}{2\pi} \exp\left[-\frac{1}{2}(x_1'^2 \mathrm{e}^{2r} + x_2'^2 \mathrm{e}^{-2r})\right]\ . \tag{4.47}$$

The contour of the Wigner function given by $Q = 1$ is

$$\frac{x_1'^2}{\mathrm{e}^{-2r}} + \frac{x_1'^2}{\mathrm{e}^{2r}} = 1 \tag{4.48}$$

which is an ellipse with the length of the major and minor axes given by $\mathrm{e}^{\boldsymbol{r}}$ and $\mathrm{e}^{-\boldsymbol{r}}$, respectively (Fig. 2.1b).

c) Number State

The Wigner function for a number state $|n\rangle$ is

$$W(x_1,x_2) = \frac{2}{\pi}(-1)^n L_n(4r^2)\mathrm{e}^{-2r^2} \tag{4.49}$$

where $r^2 = x_1^2 + x_2^2$, and $L_n(x)$ is the Laguerre polynomial. This Wigner function is clearly negative.

The Wigner function gives direct symmetrically-ordered moments such as those arising in the calculation of the variances of quadrature phases.

4.2.3 Q Function

An alternative function is the diagonal matrix elements of the density operator in a pure coherent state

$$Q(\alpha) = \frac{\langle\alpha|\rho|\alpha\rangle}{\pi} \geq 0\,. \tag{4.50}$$

This is clearly a non-negative function since the density operator is a positive operator. It is also a bounded function

$$Q(\alpha) < \frac{1}{\pi}\,.$$

Writing the distribution in terms of the real variables

$$x_1 = \alpha + \alpha^*, \quad x_2 = -\mathrm{i}(\alpha - \alpha^*)$$

the covariance matrix relation is

$$C(X_1,X_2) = C_Q(x_1,x_2) - I\,.$$

The Q function may be expressed as the Fourier transform of the antinormally-ordered characteristic function $\chi_\mathrm{A}(\eta)$

$$\chi_\mathrm{A}(\eta) = \mathrm{Tr}\{\rho \mathrm{e}^{-\eta^* a}\mathrm{e}^{\eta a^\dagger}\} = \int \frac{\mathrm{d}^2\alpha}{\pi}\langle\alpha|\mathrm{e}^{\eta a^\dagger}\rho\mathrm{e}^{-\eta^* a}|\alpha\rangle = \int \mathrm{e}^{\eta\alpha^* - \eta^*\alpha}Q(\alpha)\mathrm{d}^2\alpha\,. \tag{4.51}$$

Thus $Q(\alpha)$ is the inverse Fourier transform

$$Q(\alpha) = \frac{1}{\pi^2}\int \mathrm{e}^{\alpha\eta^* - \alpha^*\eta}\chi_\mathrm{A}(\eta)\mathrm{d}^2\eta\,. \tag{4.52}$$

The relation between the $P(\alpha)$ and the $Q(\alpha)$ follows from

$$Q(\alpha) = \frac{\langle\alpha|\rho|\alpha\rangle}{\pi} = \frac{1}{\pi}\int P(\beta)|\langle\alpha|\beta\rangle|^2\mathrm{d}^2\beta = \frac{1}{\pi}\int P(\beta)\mathrm{e}^{-|\alpha-\beta|^2}\mathrm{d}^2\beta\,. \tag{4.53}$$

That is, the Q function like the Wigner function is a Gaussian convolution of the P function. However, it is convoluted with a Gaussian which has $\sqrt{2}$ times the width of the Wigner function which accounts for the rather more well-behaved properties.

The Q function is convenient for evaluating anti-normally-ordered moments

$$\langle a^n a^{\dagger m}\rangle = \int \alpha^n \alpha^{*m} Q(\alpha,\alpha^*)\mathrm{d}^2\alpha \,. \tag{4.54}$$

The covariance matrix $C_Q(\alpha,\ \alpha^*)$ defined by the moments of α and α^* over $Q(\alpha,\ \alpha^*)$ is related to the covariance matrix $C(a,\ a^\dagger)$ defined by (4.23) by

$$C(a,a^\dagger) = C_Q(\alpha,\alpha^*) - \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{4.55}$$

The Q function has the advantage of existing for states where no P function exists and unlike the Wigner or P function is always positive. The Q functions for a coherent state and a number state are easily obtained.

For a coherent state $|\beta\rangle$ the Q function is

$$Q(\alpha) = \frac{|\langle\alpha|\beta\rangle|^2}{\pi} = \frac{\mathrm{e}^{-|\alpha-\beta|^2}}{\pi} . \tag{4.56}$$

For a number state $|n\rangle$ the Q function is

$$Q(\alpha) = \frac{|\langle\alpha|n\rangle|^2}{\pi} = \frac{|\alpha|^{2n}\mathrm{e}^{-|\alpha|^2}}{\pi n!} . \tag{4.57}$$

The Q function for a squeezed state $|\alpha,\ r\rangle$ is defined as

$$Q(\beta,\beta^*) = \frac{1}{\pi}|\langle\beta|D(\alpha)S(r)|0\rangle|^2 . \tag{4.58}$$

This is a multivariate Gaussian distribution and may be written in terms of the quadrature phase variables x_1 and x_2 as

$$Q(x_1,x_2) = \frac{1}{4\pi^2\cosh r}\exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{x}_0)^{\mathrm{T}}C^{-1}(\boldsymbol{x}-\boldsymbol{x}_0)\right] \tag{4.59}$$

where

$$\begin{aligned} \boldsymbol{x}_0 &= 2(\mathrm{Re}\{\alpha\},\ \mathrm{Im}\{\alpha\}), \\ \boldsymbol{x} &= (\boldsymbol{x}_1,\boldsymbol{x}_2), \\ C &= \begin{pmatrix} \mathrm{e}^{-2r}+1 & 0 \\ 0 & \mathrm{e}^{2r}+1 \end{pmatrix}. \end{aligned}$$

The Q function for a squeezed state is shown in Fig. 4.1.

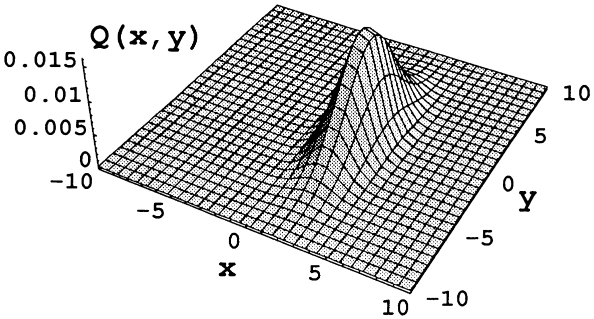

Fig. 4.1 Q function for a squeezed state with coherent amplitude $\alpha = 2.0,\ r = 1.0$

4.2.4 R Representation

Any density operator ρ may be represented in a unique way by means of a function of two complex variables $R(\alpha^*,\ \beta)$ which is analytic throughout the finite α^* and β planes. The function R is given explicitly as

$$R(\alpha^*,\beta) = \langle\alpha|\rho|\beta\rangle \exp[(|\alpha|^2 + |\beta|^2)/2] \,. \tag{4.60}$$

We may express the density operator in terms of $R(\alpha^*,\ \beta)$

$$\rho = \frac{1}{\pi^2}\int |\alpha\rangle R(\alpha^*,\beta)\langle\beta| \mathrm{e}^{-(|\alpha|^2+|\beta|^2)/2} \mathrm{d}^2\alpha\, \mathrm{d}^2\beta \,. \tag{4.61}$$

The normalization condition

$$\mathrm{Tr}\{\rho\} = 1$$

implies

$$\int \langle\gamma|\alpha\rangle\langle\alpha|\rho|\beta\rangle\langle\beta|\gamma\rangle \frac{\mathrm{d}^2\alpha}{\pi}\frac{\mathrm{d}^2\beta}{\pi}\frac{\mathrm{d}^2\gamma}{\pi} = 1 \,. \tag{4.62}$$

Interchanging the scalar products and performing the integrations over β and γ we arrive at the result

$$\frac{1}{\pi}\int \langle\alpha|\rho|\alpha\rangle \mathrm{d}^2\alpha = 1 \tag{4.63}$$

which gives the normalization condition on R

$$\frac{1}{\pi}\int R(\alpha^*,\alpha)\mathrm{e}^{-|\alpha|^2}\mathrm{d}^2\alpha = 1 \,. \tag{4.64}$$

The function $R(\alpha^*,\ \beta)$ is analytic in α^* and β (and therefore non-singular) and is by definition non-positive. It has a normalization that includes a Gaussian weight factor. For these reasons it cannot have a Fokker–Planck equation or any direct interpretation as a quasiprobability. Nevertheless, the existence of this representation does demonstrate that a calculation of normally-ordered observables for any ρ is possible with a non-singular representation.

4.2.5 Generalized P Representations

Another representation which like the R representation uses an expansion in non-diagonal coherent state projection operators was suggested by *Drummond* and *Gardiner* [5]. The representation is defined as follows

$$\rho = \int_D \Lambda(\alpha,\beta)P(\alpha,\beta)\mathrm{d}\mu(\alpha,\beta) \tag{4.65}$$

where

$$\Lambda(\alpha,\beta) = \frac{|\alpha\rangle\langle\beta^*|}{\langle\beta^*|\alpha\rangle}$$

and $\mathrm{d}\mu(\alpha,\ \beta)$ is the integration measure which may be chosen to define different classes of possible representations and D is the domain of integration. The projection operator $\Lambda(\alpha,\ \beta)$ is analytic in α and β. It is clear that the normalization condition on ρ leads to the following normalization condition on $P(\alpha,\ \beta)$

$$\int_D P(\alpha,\beta)\mathrm{d}\mu(\alpha,\beta) = 1\ . \tag{4.66}$$

Thus, the $P(\alpha,\ \beta)$ is normalisable and we shall see in Chap. 6 that it gives rise to Fokker–Planck equations. The definition given by (4.65) leads to different representations depending on the integration measure.

Useful choices of integration measure are

1. Glauber–Sudarshan P Representation

$$\mathrm{d}\mu(\alpha,\beta) = \delta^2(\alpha^* - \beta)\mathrm{d}^2\alpha\,\mathrm{d}^2\beta\ . \tag{4.67}$$

This measure corresponds to the diagonal Glauber–Sudarshan P representation defined in (4.8).

2. Complex P Representation

$$\mathrm{d}\mu(\alpha,\beta) = \mathrm{d}\alpha\ \mathrm{d}\beta\ . \tag{4.68}$$

Here $(\alpha,\ \beta)$ are treated as complex variables which are to be integrated on individual contours $C,\ C'$. The conditions for the existence of this representation are discussed in the appendix. This particular representation may take on complex values so in no sense can it have any physical interpretation as a probability distribution. However, as we shall see it is an extremely useful representation giving exact results for certain problems and physical observables such as all the single time correlation functions.

We shall now give some examples of the complex P representation.

(a) Coherent State $|\gamma_0\rangle$

Consider a density operator with an expansion in coherent states as

$$\rho = \iint_{DD'} \rho(\alpha,\beta)|\alpha\rangle\langle\beta^*|\mathrm{d}^2\alpha\,\mathrm{d}^2\beta\,. \tag{4.69}$$

Using the residue theorem

$$\rho = -\frac{1}{4\pi^2}\iint_{DD'} \rho(\alpha,\beta)\langle\beta^*|\alpha\rangle\left[\oint\oint_{CC'} \frac{\Lambda(\alpha',\beta')\mathrm{d}\alpha'\,\mathrm{d}\beta'}{(\alpha-\alpha')(\beta-\beta')}\right]\mathrm{d}^2\alpha\,\mathrm{d}^2\beta\,. \tag{4.70}$$

Exchanging the order of integration we see the complex P function is

$$P(\alpha,\beta) = -\frac{1}{4\pi^2}\iint_{DD'} \rho(\alpha',\beta')\langle\beta'^*|\alpha'\rangle\frac{\mathrm{d}^2\alpha'\mathrm{d}^2\beta'}{(\alpha-\alpha')(\beta-\beta')}\,. \tag{4.71}$$

Thus for a coherent state $|\gamma_0\rangle$

$$P(\alpha,\beta) = -\frac{1}{4\pi^2(\alpha-\gamma_0)(\beta-\gamma_0^*)}\,. \tag{4.72}$$

Examples of complex P functions for nonclassical fields where the Glauber–Sudarshan P function would be highly singular are given below.

(b) Number State $|n\rangle$

$$P(\alpha,\beta) = -\frac{1}{4\pi^2}\mathrm{e}^{\alpha\beta}\frac{n!}{(\alpha\beta)^{n+1}}\,. \tag{4.73}$$

This may be proved as follows. Using

$$\langle\alpha|\beta\rangle = \mathrm{e}^{\alpha^*\beta-|\alpha|^2/2-|\beta|^2/2} \tag{4.74}$$

and

$$|\alpha\rangle = \sum\frac{\mathrm{e}^{-|\alpha|^2/2}\alpha^n|n\rangle}{n!^{1/2}}\,, \tag{4.75}$$

we may write ρ as

$$\rho = \int P(\alpha,\beta)\sum\frac{|n'\rangle\langle m'|}{(n'!)^{1/2}(m'!)^{1/2}}\,\mathrm{e}^{-\alpha\beta}(\alpha^{n'}\beta^{m'})\mathrm{d}\alpha\,\mathrm{d}\beta\,. \tag{4.76}$$

Substituting (4.73) for $P(\alpha,\ \beta)$

$$\rho = -\frac{1}{4\pi^2}\sum\frac{(n!)^2}{(n'!)^{1/2}(m'!)^{1/2}}\int\alpha^{-(n+1-n')}\beta^{-(n+1-m')}|n'\rangle\langle m'|\mathrm{d}\alpha\,\mathrm{d}\beta\,. \tag{4.77}$$

Choosing any contour of integration encircling the origin and using Cauchy's theorem

$$\begin{aligned}\frac{1}{2\pi i}\oint dz\, z^n &= 0 \quad \text{if } n \geq 0,\\ &= 1 \quad \text{if } n = -1,\\ &= 0 \quad \text{if } n < -1\,. \end{aligned} \tag{4.78}$$

We find

$$\rho = |n\rangle\langle n|\,.$$

(c) Squeezed State $|\gamma,\ r\rangle$

The complex P representation for a squeezed state is

$$P(\alpha,\beta) = N\exp\{(\alpha-\gamma)(\beta-\gamma^*) + \coth r[(\alpha-\gamma)^2 + (\beta-\gamma^*)^2]\}\,. \tag{4.79}$$

This may be normalized by integrating along the imaginary axis for r real. The resulting normalization for this choice of contour is

$$N = -\frac{1}{2\pi\sinh r}\,.$$

As an example of the use of the complex P representation we shall consider the photon counting formula given by (3.107). Using the diagonal coherent-state representation for ρ we may write the photon counting probability $P_m(T)$ as

$$P_m(T) = \int d^2 z P(z)\frac{(|z|^2\mu(T))^m}{m!}\exp[-|z|^2\mu(T)]\,. \tag{4.80}$$

An appealing feature of this equation is that $P_m(T)$ is given by an averaged Poisson distribution with $P(z)$ in the role of a probability distribution over the complex field amplitude. It is a close analogue of the classical expression (3.96). We know however that $P(z)$ is not a true probability distribution and may take on negative values, and this may cause some anxiety over the validity of (4.80). In such cases we may consider a simple generalization of (4.80) by using the complex P representation for ρ. The photocount probability is then given by

$$P_m(T) = \int_{CC'} dz\, dz'\; P(z,z')\frac{(zz'\mu(T))^m}{m!}\exp[-zz'\mu(T)]\,. \tag{4.81}$$

We shall demonstrate the use of this formula to calculate $P_m(T)$ for states for which no well behaved diagonal P distribution exists.

a) Number State

For a number state with density operator $\rho = |n\rangle\langle n|$ we have

$$P(z,z') = -\frac{1}{4\pi^2}\exp(zz')n!(zz')^{-n-1} \tag{4.82}$$

and the contours C and C' enclose the origin. Substituting (4.81) for $m > n$ the integrand contains no poles and $P_m(T) = 0$, while for $m < n$ poles of order $n - m + 1$ contribute in each integration and we obtain the result of (3.112)

$$P_m(T) = \sum_{n=m}^{\infty} \binom{n}{m} [\mu(T)]^m [1-\mu(T)]^{n-m} \,. \tag{4.83}$$

b) Squeezed State

For a squeezed state with density operator $\rho = |\gamma,\ r\rangle\langle\gamma,\ r|$ where γ and r are taken to be real, we have

$$\begin{aligned} P(z,z') = -\frac{1}{2\pi}(\sinh r)^{-1} \exp\{&(z-\gamma)(z'-\gamma) + \coth r[(z-\gamma)^2 \\ &+ (z'-\gamma)^2]\} \end{aligned} \tag{4.84}$$

and the contours C and C' are along the imaginary axes in z and z' space, respectively. Performing the integration in (4.81) gives the formula, see (3.112),

$$P_m(T) = \sum_{n=m}^{\infty} \binom{n}{m} [\mu(T)]^m [1-\mu(T)]^{n-m} P_n \tag{4.85}$$

with

$$P_n = (n!\ \cosh r)^{-1} \exp[-\gamma^2 e^{2r}(1+\tanh r)](\tanh r)^n H_n^2\left(\frac{\gamma e^r}{\sqrt{\sinh 2r}}\right)$$

where the $H_n(x)$ are Hermite polynomials. This agrees with the result of a derivation using the number state representation (3.109) when we recognise that $P_n = |\langle n|\gamma,\ r\rangle|^2$.

4.2.6 Positive P Representation

The integration measure is chosen as

$$\mathrm{d}\mu(\alpha,\beta) = \mathrm{d}^2\alpha\, \mathrm{d}^2\beta \,. \tag{4.86}$$

This representation allows $\alpha,\ \beta$ to vary independently over the whole complex plane. It was proved in [5] that $P(\alpha,\ \beta)$ always exists for a physical density operator and can always be chosen positive. For this reason we call it the positive P representation. $P(\alpha,\ \beta)$ has all the mathematical properties of a genuine probability. It may also have an interpretation as a probability distribution [6]. It proves a most useful representation, in particular, for problems where the Fokker–Planck equation in other representations may have a non-positive definite diffusion matrix. It may be shown that provided any Fokker–Planck equation exists for the time development in the Glauber–Sudarshan representation, a corresponding Fokker–Planck equation exists with a positive semidefinite diffusion coefficient for the positive P representation.

Exercises

4.1 Show that if a field with the P representation $P_i(\alpha)$ is incident on a 50/50 beam splitter the output field has a P representation given by $P_0(\alpha) = 2P_i(\sqrt{2}\alpha)$.

4.2 Show that the Wigner function may be written, in terms of the matrix elements of ρ in the eigenstates of X_1, as

$$W(x_1, x_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{e}^{-\mathrm{i}xx_2} \langle x_1 + x|\rho|x_1 - x\rangle$$

where $x_1 = \alpha + \alpha^*$ and $x_2 = -\mathrm{i}(\alpha - \alpha^*)$.

4.3 The complex P representation for a number state $|n\rangle$ is

$$P(\alpha, \beta) = -\frac{1}{4\pi^2} \mathrm{e}^{\alpha\beta} \frac{n!}{(\alpha\beta)^{n+1}} \,.$$

Show that

$$\langle a^\dagger a\rangle = \oint \mathrm{d}\alpha \, \mathrm{d}\beta \; \alpha\beta \; P(\alpha, \, \beta) = n \,.$$

4.4 Use the complex P representation for a squeezed state $|\gamma, \, \boldsymbol{r}\rangle$ with γ and r both real, to show that the photon number distribution for such a state is

$$P_n = (n! \; \cosh r)^{-1} \exp[-\gamma^2 \mathrm{e}^{2r}(1 + \tanh r)](\tanh r)^n H_n^2 \left(\frac{\gamma \mathrm{e}^r}{\sqrt{\sinh 2r}}\right) .$$

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